

On the extreme value statistics of normal random matrices

R. Ebrahimi¹ and S. Zohren²

¹ Department of Physics, Pontifica Universidade Católica, Rio de Janeiro, Brazil

² Department of Materials, Advanced Research Computing, University of Oxford, UK

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Abstract. In this paper we extend the approach of orthogonal polynomials for extreme value calculations of Hermitian random matrices, developed by Nadal and Majumdar [1102.0738], to normal random matrices and 2D Coulomb gases in general. Firstly, we show that this approach provides an alternative derivation of results in the literature. More precisely, we show convergence of the rescaled eigenvalue with largest modulus of a Ginibre ensemble to a Gumbel distribution, as well as universality for an arbitrary radially symmetric potential. Secondly, it is shown that this approach can be generalised to obtain convergence of the eigenvalue with smallest modulus and its universality for ring distributions. Furthermore, the here presented techniques can be used to compute finite N expressions of the above distributions, which is important for practical applications given the slow convergence.

1. Introduction

Complex non-Hermitian random matrices were first investigated by Ginibre [1], who analysed matrices whose elements are independent and identically distributed (real or complex) Gaussian variables. He showed that the eigenvalue density $\mu_N(z)$ of such N dimensional random matrices in the limit $N \rightarrow \infty$ is given by the well-known circular law ‡

$$\mu(z) := \lim_{N \rightarrow \infty} \mu_N(z) = \frac{1}{\pi} \mathbb{I}_{\{|z| \leq 1\}}. \quad (1)$$

In other words, the eigenvalues are distributed uniformly over the disk of radius 1 in the complex plane (see Figure 1 (Left)).

In recent years there has been an increasing interest in non-Hermitian random matrices (see e.g. [2, 3, 4, 5, 6] for some examples as well as Chapter 18 of [7] for a review). Random matrices where all elements are chosen independently from some distribution are generally referred to as Wigner matrices. The Gaussian case discussed above is a special case since it allows for a Coulomb gas formulation describing the joint eigenvalues distribution in terms of an attractive Gaussian potential and a logarithmic, Coulomb repulsion [1]. In general, Wigner random matrices do not allow for such a formulation due to the lack of symmetries. One closely related ensemble is that of normal random matrices [8] where one introduces an additional symmetry that the matrices are normal (they commute with their adjoint). This symmetry allows for a Coulomb gas formulation with arbitrary potential.

Normal random matrices have been studied intensively during the last years [9, 10, 11, 12]. In this work we are interested in the extreme value statistics of normal random matrices with general radially symmetric potential. As the eigenvalues can spread over the complex plane (within the support), they are less correlated than in the unitary or orthogonal case. This is reflected by the extreme value statistics. In the above example, considering the distribution of the eigenvalue with the largest modulus, it is known [13, 14] that the rescaled distribution is given by a Gumbel distribution. Recall that the Gumbel distribution is one of three families which describe the extreme value statistics of independent and identically distributed random variables (Weibull, Gumbel, Fréchet) [15, 16, 17]. While the eigenvalues of non-Hermitian ensembles are not independent, their correlation is much weaker than for Hermitian ensembles. Indeed for Hermitian ensembles the eigenvalues are strongly correlated and the distribution of the largest eigenvalue is given by the Tracy-Widom distribution [18, 19, 20].

Outline and what's new. After a brief introduction to normal normal matrices and their relation to 2D Coulomb gases in Section 2, we extend in Section 3 the orthogonal polynomials approach to extreme value statistics of random matrices from the Hermitian

‡ The eigenvalue density is formally defined as $\int_A \mu_N(z) d^2z = \frac{1}{N} \mathbb{E}_N(\#\{\text{eigenvalues in } A\})$ for any $A \subset \mathbb{C}$, where $\mathbb{E}_N(\cdot)$ refers the expectation value with respect to the joint probability density function \mathbb{P}_N over the N eigenvalues.

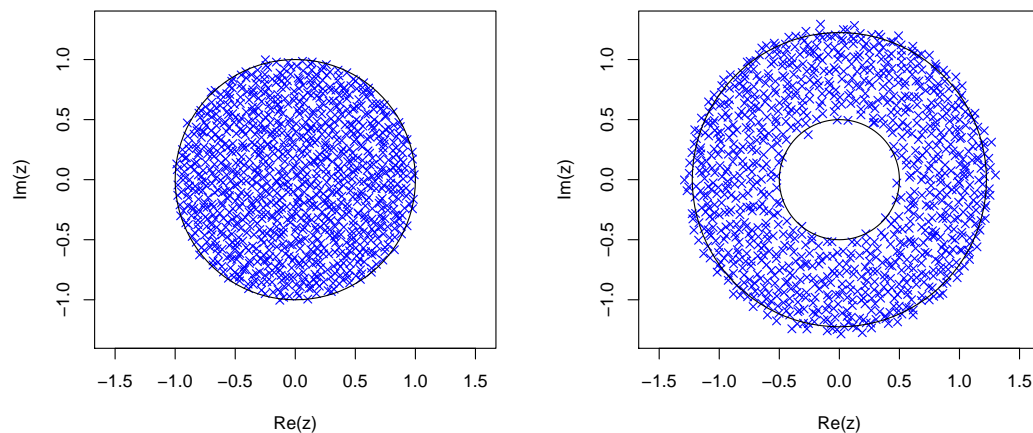


Figure 1. Left: Distribution of eigenvalues of a 1000×1000 Ginibre matrix equivalent to a Coulomb gas with Gaussian potential $V(r) = r^2$. The eigenvalues density is given by the circular law. Right: Distribution of eigenvalues of a 1000×1000 normal random matrix with potential $V(r) = r^2 - r$ obtained using Monte-Carlo methods. The eigenvalue density has support on a ring.

case [21] to non-Hermitian random matrices. In Section 4 and 5 we use this approach to show convergence of probability distribution of the rescaled eigenvalue with largest modulus of normal random matrices to a Gumbel distribution as well as universality of this result. This provides a simplified, alternative derivation of result in [13] and [14] for Coulomb gases. In Section 6 we show that our approach can also be used to show convergence and universality of the distribution of the eigenvalue with smallest modulus at the inner edge of a ring distribution. We conclude in Section 7 where we also discuss how the here presented approach can be used to obtain finite N corrections of such extreme value statistics of normal random matrices and 2D Coulomb gases. At various places in this article analytical results are complemented with numerical results. The methods to obtain those numerical results are explained in Appendix A.

2. Normal random matrices and Coulomb gases

Normal random matrices were introduced in [8] and are defined through the measure

$$\mathbb{P}_N(M) dM = \frac{1}{N! Z_N} e^{-N \operatorname{tr} V(M)} dM, \quad (2)$$

where M are complex $N \times N$ dimensional matrices satisfying the constraint $[M, M^\dagger] = 0$. Here $dM = \prod_{ij} d\Re M_{ij} d\Im M_{ij}$ is the so-called Haar measure and the normalisation

$$Z_N = \frac{1}{N!} \int e^{-N \operatorname{tr} V(M)} dM \quad (3)$$

is also referred to as the partition function.

The eigenvalue statistics of normal random matrices is in many ways similar to that of general non-Hermitian matrices. However, normal random matrices are computationally much more feasible due to their symmetry. In particular, the normality condition $[M, M^\dagger] = 0$ implies that there exists a unitary matrix which simultaneously diagonalises M and M^\dagger . This allows one to employ Coulomb gas techniques and find the joint probability measure of the complex eigenvalues, given by

$$\mathbb{P}_N(z_1, \dots, z_N) = \frac{1}{N! Z_N} \prod_{1 \leq j < k \leq N} (z_j - z_k)^2 e^{-N \sum_{i=1}^N V(z_i)}. \quad (4)$$

Here the term $\Delta(\{z\}) := \prod_{1 \leq j < k \leq N} (z_j - z_k)$ is called the Vandermonde determinant. Writing

$$\mathbb{P}_N(z_1, \dots, z_N) = \frac{e^{-N^2 S_{\text{eff}}(z_1, \dots, z_N)}}{N! Z_N} \quad (5)$$

with

$$S_{\text{eff}}(z_1, \dots, z_N) = \frac{1}{N} \sum_{i=1}^N V(z_i) - \frac{2}{N^2} \sum_{1 \leq j < k \leq N} \log |z_j - z_k| \quad (6)$$

we can interpret the effective model as that of a gas of N particles in a potential $V(z)$ experiencing a logarithmic repulsion. This is equivalent to a Coulomb repulsion in 2D, thus the name Coulomb gas.

As mentioned above, the (complex or real) Ginibre ensemble where all entries are (complex or real) Gaussian random variables also has a joint probability density function of its eigenvalues given by (4) with Gaussian potential, however, for arbitrary Wigner matrices there does not exist an analogous relation. Thus it is interesting to study normal random matrices or Coulomb gases with general potential $V(z)$.

In this article we are only interested in radially symmetric potentials

$$V(z) \equiv V(r), \quad z = r e^{i\phi}. \quad (7)$$

In this case one can use saddle point techniques or more mathematically results from the theory of harmonic analysis [22] to find the eigenvalue density for a given radially symmetric potential. More precisely, given the condition that

$$\lim_{r \rightarrow \infty} (V(r) - \log r^2) = \infty \quad (8)$$

and $r V'(r)$ increasing in \mathbb{R}^+ or $V'(r) > 0$ and V convex in \mathbb{R}^+ one finds that the eigenvalue density is given by [22]

$$\mu(z) d^2 z = \frac{1}{4\pi} \frac{d}{dr} (r V'(r)) \mathbb{I}_{\{a_- \leq r \leq a_+\}} dr d\phi. \quad (9)$$

We see that the eigenvalue density has support on a ring with inner radius a_- and outer radius a_+ (or a disk in the case where $a_- = 0$), where the radii are given by

$$V'(a_-) = 0 \quad (10)$$

and a_+ is the smallest solution to following equation

$$a_+ V'(a_+) = 2. \quad (11)$$

We see that the eigenvalue density either has topology of a disk or of an annulus. This is the famous one ring theorem [6, 22, 23].

As an example, for the Gaussian normal ensemble with potential $V(r) = r^2$, we have $a_- = 0$ and $a_+ = 1$, and the eigenvalues spread over the unit disk in the complex plane. We see that the eigenvalue density is identical to that of the Ginibre ensemble, i.e., the circular law (see Figure 1 (Left)). Another easy but more interesting example is the potential $V(r) = r^2 + sr$. Here for $s > 0$ the topology of the support is a disk while for $s < 0$ the topology is that of an annulus (see Figure 1 (Right)). Interesting statistics can be computed for the transition when the topology changes from disk to annulus (see for example [24]).

Besides saddle point analysis, another powerful technique employed in random matrix theory in general and in this article in particular is the method of orthogonal polynomials. The aim of the orthogonal polynomials approach is to define a set of polynomials which are orthogonal with respect to a weight $w(z)$, i.e.,

$$\langle p_n | p_m \rangle_w := \int w(z) p_n(z) \overline{p_m(z)} d^2 z = h_n \delta_{nm} \quad (12)$$

where in our case $w(z) = e^{-NV(z)}$. Having found such a set of polynomials one can use them to evaluate many quantities in random matrix theory, in particular, the partition function can be obtained as

$$Z_N = \prod_{n=0}^{N-1} h_n, \quad (13)$$

where h_n are the normalisation constants of the orthogonal polynomials as defined in (12).

For the case of radially symmetric potential the orthogonal polynomials become very simple. In fact, for any weight $w(z) = w(|z|)$ it can be readily checked that they are simply given by the monomials

$$p_n(z) = z^n. \quad (14)$$

Thus one has

$$h_n = 2\pi \int_0^\infty r^{2n+1} w(r) dr = 2\pi \int_0^\infty r^{2n+1} e^{-NV(r)} dr. \quad (15)$$

For the case of the Gaussian potential with $V(r) = r^2$ the normalisation constants are given by $h_n = \pi N^{-n-1} \Gamma(n+1)$.

3. Orthogonal polynomial approach for extreme value statistics of random matrices

One of the most well-known examples of extreme value statistics of random matrices is the Tracy-Widom distribution [18, 19, 20] originally obtained for the GUE. The orthogonal polynomial approach to extreme value statistics of random matrices was first introduced in [21] to provide an easy derivation of the Tracy-Widom distribution

for the GUE. It has then been extended to arbitrary potentials and multi-critical points [25] as well as large deviations [26]. Here we show how one can use the same formulation for normal random matrices with complex eigenvalues. More precisely we are interested in the statistics of the complex eigenvalue z_{max} with the largest modulus. The basic idea underlying this approach is simple. Let us define a partition function $Z_N(\mathcal{D})$ where the eigenvalues are restricted to a domain $\mathcal{D} \subseteq \mathbb{C}$,

$$Z_N(\mathcal{D}) = \frac{1}{N!} \int \prod_{1 \leq j < k \leq N} (z_j - z_k)^2 e^{-N \sum_{i=1}^N V(z_i)} \prod_j \mathbb{I}_{z_j \in \mathcal{D}} d^2 z_j, \quad (16)$$

where \mathbb{I}_C is the indicator function which is 1 if C is true and 0 otherwise. Then $Z_N(\mathbb{C}) \equiv Z_N$, i.e., when integrating over the entire domain we recover the conventional partition function. Given the above, it follows that

$$\mathbb{P}_N(\text{all eigenvalues in } \mathcal{D}) = \frac{Z_N(\mathcal{D})}{Z_N}. \quad (17)$$

We can use this relation to express extreme value statistics in terms of $Z_N(\mathcal{D})$. More precisely, let us choose $\mathcal{D} = \{z : |z| \leq y\}$ and denote as shorthand $Z_N(y) := Z_N(\{z : |z| \leq y\})$ (and thus $Z_N(\infty) = Z_N$), then the cumulative probability function of the eigenvalue with the largest modulus is given by

$$F_N(y) := \mathbb{P}_N(|z_{max}| \leq y) = \mathbb{P}_N(|z_1| \leq y, \dots, |z_N| \leq y) = \frac{Z_N(y)}{Z_N(\infty)}. \quad (18)$$

Following the analogous approach for the GUE [21], the key idea is now to define orthogonal polynomials for the partition function $Z_N(y)$. In particular, we can find the orthogonal polynomials such that

$$\int e^{-N V(z)} \mathbb{I}_{\{|z| \leq y\}} p_n(z; y) \overline{p_m(z; y)} d^2 z = h_n(y) \delta_{nm}. \quad (19)$$

We have

$$Z_N(y) = \prod_{n=0}^{N-1} h_n(y) \quad (20)$$

and thus

$$F_N(y) = \mathbb{P}_N(|z_{max}| \leq y) = \prod_{n=0}^{N-1} \frac{h_n(y)}{h_n(\infty)}. \quad (21)$$

As above, we are interested in the case of radially symmetric potentials $V(z) = V(|z|)$. In this case, we observe that (19) is equivalent to (12) with a weight $w(z) \equiv w(|z|; y) = e^{-N V(|z|)} \mathbb{I}_{\{|z| \leq y\}}$. Since $w(z)$ is radially symmetric, we know from the previous section that the orthogonal polynomials are simply given by the monomials $p_n(z; y) := z^n$. Therefore, using the first equation in (15) we obtain

$$h_n(y) = 2\pi \int_0^y r^{2n+1} e^{-N V(r)} dr, \quad (22)$$

where the indicator function in the weight reduces the integration range of r to $[0, y]$.

Once N becomes large the probability density $f_N(y) = F'_N(y)$ will be more and more centred around the edge of the support, a_+ . More precisely, one expects $(|z_{\max}| - a_+) \sim N^{-1/2}$, thus to obtain convergence to a limiting distribution one has to rescale y to a “continuum” variable Y ,

$$y = a_+ + CN^{-1/2}(\alpha_N + \beta_N Y), \quad (23)$$

where C is a constant and α_N, β_N are slowly varying functions. The constant C is chosen merely for convenience and could equally be absorbed in α_N and β_N . Since $y \geq 0$, we have $Y \geq -\alpha_N/\beta_N$. For the limiting distribution to have support on $Y \in (-\infty, +\infty)$, we thus require $\alpha_N/\beta_N \rightarrow \infty$ as $N \rightarrow \infty$. This is enough to obtain the extreme value statistics related to the eigenvalue with the largest modulus, as we will show in the next section for the Gaussian potential.

4. The Gaussian potential

From the results of the previous section, more precisely (21) and (22), it follows that the cumulative probability function of the eigenvalue with largest modulus for the Gaussian random normal matrix is given by

$$\begin{aligned} F_N(y) &= \mathbb{P}_N(|z_{\max}| \leq y) = \prod_{n=0}^{N-1} \frac{\int_0^y r^{2n+1} e^{-Nr^2} dr}{\int_0^\infty r^{2n+1} e^{-Nr^2} dr} \\ &= \prod_{n=0}^{N-1} \frac{\gamma(n+1, Ny^2)}{\Gamma(n+1)} = \prod_{n=0}^{N-1} P(n+1, Ny^2) \end{aligned} \quad (24)$$

in which $P(a, z) := \gamma(a, z)/\Gamma(a)$ is the regularised incomplete gamma function.

One can try to reduce the large N behaviour of (24) using the asymptotic expansions of the regularised incomplete gamma function, where both arguments become large, however, it is more instructive to calculate the asymptotic behaviour from first principles using a saddle point evaluation. In particular, such a derivation will generalise to arbitrary potentials, where we cannot find a closed form expression such as (24).

To perform the asymptotic expansion using a saddle point evaluation we start by writing

$$h_n(y) = 2\pi \int_0^y e^{Nf(r)} dr, \quad \text{with} \quad f(r) = \frac{2n+1}{N} \log r - r^2. \quad (25)$$

We can now expand $f(r)$ around its maximum at $r_0 = \sqrt{(2n+1)/(2N)}$ as

$$f(r) = f(r_0) - \frac{1}{2}|f''(r_0)|(r - r_0)^2 + \mathcal{O}((r - r_0)^3), \quad (26)$$

The integration in (22) then reduces to Gaussian integrals over finite intervals, yielding error functions:

$$\frac{h_n(y)}{h_n(\infty)} = \frac{\text{erf}(\sqrt{Nb}r_0) + \text{erf}(\sqrt{Nb}(y - r_0))}{1 + \text{erf}(\sqrt{Nb}r_0)}, \quad (27)$$

where $b \equiv |f''(r_0)/2| = 2$ for the Gaussian potential.

Now using the Euler-Maclaurin summation formula, we look at the first two leading order terms in (21) when both N and n become large, while their ratio is given by $\frac{n}{N} =: \xi \in [0, 1]$. One obtains

$$F_N(y) = N \int_0^1 g(\xi; y) d\xi + \frac{1}{2}[g(1; y) - g(0; y)] + \mathcal{O}(N^{-1}), \quad (28)$$

where we defined

$$g(\xi; y) := \log \left(\frac{\operatorname{erf}(\sqrt{Nb} r_0(\xi)) + \operatorname{erf}(\sqrt{Nb}(y - r_0(\xi)))}{1 + \operatorname{erf}(\sqrt{Nb} r_0(\xi))} \right) \quad (29)$$

in which $r_0(\xi) = \sqrt{\xi} + \mathcal{O}(N^{-1})$. Recall the asymptotic expansion of the error function

$$\operatorname{erf}(z) \equiv \frac{2}{\sqrt{\pi}} \int_0^z e^{-x^2} dx \sim 1 - \frac{e^{-z^2}}{\sqrt{\pi}z}, \quad z \rightarrow \infty \quad (\Re(z) > 0). \quad (30)$$

Note that the argument of the first error function in the denominator of (29) is not necessarily large if ξ is close to zero. However, the asymptotics of (29) is dominated by the second term

$$g(\xi; y) \sim -\frac{1}{\sqrt{\pi}} \frac{e^{-Nb(y-r_0(\xi))^2}}{\sqrt{Nb}(y-r_0(\xi))(1 + \operatorname{erf}(\sqrt{Nb} r_0(\xi)))}, \quad N \rightarrow \infty. \quad (31)$$

Note that the scaling (23) ensures that the argument of the second error function $\sqrt{Nb}(y - r_0(\xi)) \rightarrow \infty$ as $N \rightarrow \infty$ as long as the slowly varying function $\alpha_N \rightarrow \infty$. Furthermore, since $r_0(\xi) \rightarrow a_+$ as $\xi \rightarrow 1$, the dominant contribution in (31) comes from ξ close to one, in which case we have

$$g(\xi; y) \sim -\frac{1}{2\sqrt{\pi}} \frac{e^{-Nb(y-r_0(\xi))^2}}{\sqrt{Nb}(y-r_0(\xi))}, \quad N \rightarrow \infty. \quad (32)$$

Therefore, the leading term on the right hand side of (28) is given by

$$N \int_0^1 g(\xi; y) d\xi \sim -\frac{N}{2\sqrt{\pi}} \int_0^1 \frac{e^{-Nb(y-r_0(\xi))^2}}{\sqrt{Nb}(y-r_0(\xi))} d\xi. \quad (33)$$

Let us now insert the scaling relation (23). We know that α_N is a slowly varying function which diverges for $N \rightarrow \infty$. Furthermore, we require $\alpha_N/\beta_N \rightarrow \infty$ as $N \rightarrow \infty$. It will become clear later that we will have to choose $\beta_N \sim 1/\alpha_N$ and moreover that it will be convenient to choose $\beta_N = 1/(2\alpha_N)$. The factor of 2 is merely a rescaling of Y which could otherwise be determined at the end of the calculation. We thus have the scaling

$$y = a_+ + (Nb)^{-1/2} \left(\alpha_N + \frac{1}{2\alpha_N} Y \right), \quad (34)$$

where here $a_+ = 1$. Since we know that the contribution of $g(\xi; y)$ comes from ξ close to one, we introduce the following change of variable in the integration in (33),

$$\xi = 1 - \frac{(2N)^{-1/2}}{\alpha_N} X. \quad (35)$$

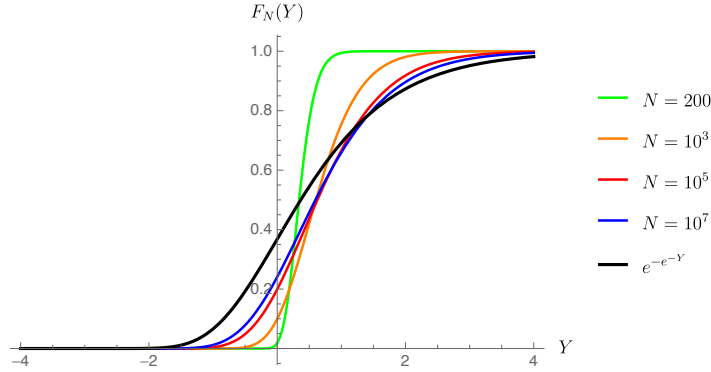


Figure 2. Shown is the function $F_N(Y)$ for increasing values of N , as well as the limiting case $F(Y) = \exp(-\exp(-Y))$. The function $F_N(Y)$ is obtained by numerically evaluating (24) with y replaced with its scaling relation (34).

For this choice one has

$$\sqrt{Nb}(y - r_0(\xi)) = \alpha_N + \frac{1}{2\alpha_N} (X + Y) + \mathcal{O}(N^{-1/2}\alpha_N^{-1}). \quad (36)$$

This relation also justifies the above choices of scalings. In particular, $Nb(y - r_0(\xi))^2 = \alpha_N^2 + (X + Y) + \dots$ is a slowly varying function plus a finite term. To get this it was necessary to choose $\beta_N \sim 1/\alpha_N$. Putting everything together, one then obtains

$$\begin{aligned} N \int_0^1 g(\xi; y) d\xi &= -\frac{\sqrt{N}}{2\sqrt{2\pi}\alpha_N^2} \int_0^{\sqrt{2N}\alpha_N} e^{-[\alpha_N^2 + (X+Y)]} dX \\ &= -\frac{\sqrt{N}e^{-\alpha_N^2}}{2\sqrt{2\pi}\alpha_N^2} \left(1 - e^{-\sqrt{2N}\alpha_N}\right) e^{-Y}. \end{aligned} \quad (37)$$

Now if we make the following choice for the slowly varying function α_N ,

$$\alpha_N^2 = \frac{1}{2} (\log N - 2 \log \log N - \log 2\pi). \quad (38)$$

Using this one arrives at

$$N \int_0^1 g(\xi; y) d\xi = -e^{-Y} + \mathcal{O}(\log \log N / \log N). \quad (39)$$

Furthermore, under this choice, for the subleading term on the right hand side of (28) we have

$$\frac{1}{2} [g(1; y) - g(0; y)] = \mathcal{O}(e^{-\alpha_N^2} \alpha_N^{-1}) = \mathcal{O}(\sqrt{\log N / N}). \quad (40)$$

The final result is thus given by

$$\begin{aligned} F(Y) &:= \lim_{N \rightarrow \infty} F_N(Y) \\ &= \lim_{N \rightarrow \infty} \mathbb{P}_N \left(|z_{\max}| \leq 1 + (Nb)^{-1/2} \left[\alpha_N + \frac{Y}{2\alpha_N} \right] \right) = e^{-e^{-Y}}, \end{aligned} \quad (41)$$

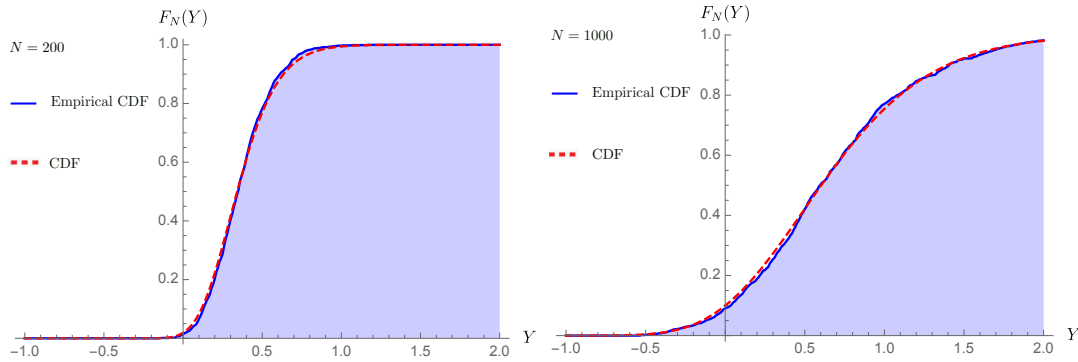


Figure 3. Comparison of the finite N expression $F_N(Y)$ as given in (24) with the empirical cumulative probability function obtained from sampling 1000 Ginibre matrices of size $N = 200$ and $N = 1000$ respectively.

which is precisely the Gumbel distribution. This provides an alternative derivation of the results in [13].

One strength of the here presented approach is that it is particularly suited for calculating finite N corrections of the limiting distribution $\lim_{N \rightarrow \infty} F_N(Y) = F(Y) = \exp(-\exp(-Y))$. This can be important in practice since the convergence to the Gumbel distribution is very slow as can be seen from the order of the subleading term in (39).

Figure 2 shows $F_N(Y)$ evaluated for increasing values of N . It is clear that for any practical applications, the limiting distribution is not a good approximation. However, the result presented in (24) provides a closed expression for $F_N(Y)$ for finite N in terms of the regularised incomplete gamma function which can easily be evaluated numerically by inserting the slowly varying function (38) into (24), as had been done in Figure 2. For large values of N , one can benefit from the fact that the sum in (24) is dominated by $n \approx N$. This is equivalent to the statement that the major contribution to the integral (33) comes from the terms with ξ close to one, as given by (35). We can also verify the correctness of this expression by comparing the result of $F_N(Y)$ with the empirical cumulative probability function obtained from sampling m representations of a Ginibre matrix of size N . This is shown in Figure 3 for $m = 1000$, with $N = 200$ and $N = 1000$ respectively.

Note that in principle the here presented techniques can also be used to compute the next-to-leading order term in (39) and (41).

5. Universality at the outer edge

In this section we consider the generalisation of the Gaussian potential to an arbitrary radially symmetric potential, $V \equiv V(r)$. The approach presented in the previous section translates directly to an arbitrary potential by simply replacing $f(r)$ in (25) with

$$f(r) = \frac{2n+1}{N} \log r - V(r) \quad (42)$$

The local extremum $r_0 = r_0(\xi)$ determined by $f'(r_0) = 0$ is given by the solution to the

following equation

$$r_0(\xi) V'(r_0(\xi)) = \frac{2n+1}{N} = 2\xi + N^{-1}. \quad (43)$$

Comparing (43) and (11) shows that in the large N limit, the outer edge and r_0 coincide, i.e., $r_0(1) = a_+$.

Next we Taylor expand $f(r)$ around its extremum at r_0 . To ensure that the extremum is indeed a maximum we compute the second derivative,

$$f''(r_0) = - \left(\frac{V'(r_0)}{r_0} + V''(r_0) \right) = - \left(\frac{1}{r} \frac{d}{dr} (r V'(r)) \right) \Big|_{r=r_0}. \quad (44)$$

Therefore, we have $f''(r_0) < 0$ and thus have a maximum, if the condition

$$r V'(r) \text{ increasing in } \mathbb{R}^+ \quad (45)$$

or equivalently

$$V'(r) > 0 \text{ and } V \text{ convex in } \mathbb{R}^+ \quad (46)$$

holds for the potential $V(r)$. Note that these conditions are the same as the ones given below (8).

Since we have shown that $f''(r_0) < 0$ under the condition (45) or (46), it then follows that (27) and (32) from the Gaussian case remain valid, where now $f''(r_0)$ in $b = |f''(r_0)|/2$ is given by (44). Therefore, the leading order term in the probability distribution of the eigenvalue with the largest modulus is given by

$$N \int_0^1 g(\xi; y) d\xi \sim -\frac{N}{2\sqrt{\pi}} \int_0^1 \frac{e^{-Nb(y-r_0(\xi))^2}}{\sqrt{Nb}(y-r_0(\xi))} d\xi, \quad N \rightarrow \infty. \quad (47)$$

As in the Gaussian case, the general scaling of y is given by (34). We thus also change the integration variable ξ in (47) as in (35) only with an additional constant $\gamma_+ > 0$,

$$\xi = 1 - \gamma_+ \frac{(Nb)^{-1/2}}{2\alpha_N} X. \quad (48)$$

The value of γ_+ will be determined in the following. We expand $r_0(\xi)$ around $\xi = 1$:

$$\begin{aligned} r_0(\xi) &= r_0(1) + dr_0(\xi)/d\xi|_{\xi=1} (\xi - 1) + \mathcal{O}((\xi - 1)^2) \\ &= a_+ + \delta_+ (\xi - 1) + \mathcal{O}((\xi - 1)^2), \end{aligned} \quad (49)$$

where δ_+ is given by

$$\begin{aligned} \delta_+ &:= dr_0(\xi)/d\xi|_{\xi=1} = \frac{2}{V'(a_+) + a_+ V''(a_+)} \\ &= 2 \left(\frac{d}{dr} (r V'(r)) \Big|_{r=a_+} \right)^{-1}. \end{aligned} \quad (50)$$

Hence, according to the constraint already imposed in (45), we have $\delta_+ > 0$ and therefore obtain

$$\sqrt{Nb} (y - r_0(\xi)) = \alpha_N + \frac{1}{2\alpha_N} (\gamma_+ \delta_+ X + Y) + \mathcal{O}(N^{-1/2} \alpha_N^{-1}). \quad (51)$$

which by choosing

$$\gamma_+ = \frac{1}{\delta_+} = \frac{1}{2} \frac{d}{dr} \left(r V'(r) \right) \Big|_{r=a_+} \quad (52)$$

reduces to (36). The rest of the derivation is identical to the Gaussian case presented in the previous section. We have thus shown universality of the distribution of the eigenvalue with the largest modulus when rescaled around the outer edge of a generic potential $V(r)$ satisfying the condition given in (45) or (46), where the limiting probability distribution is the Gumbel distribution

$$\lim_{N \rightarrow \infty} \mathbb{P}_N \left(|z_{\max}| \leq a_+ + (Nb)^{-1/2} \left[\alpha_N + \frac{Y}{2\alpha_N} \right] \right) = e^{-e^{-Y}}. \quad (53)$$

This reproduces results from [14] using our general approach.

6. Universality at the inner edge

Now we study the extreme value statistics of a normal matrix ensemble with potential $V = V(r)$ at the inner edge of the eigenvalue support for potentials where $a_- > 0$. To do so we now choose \mathcal{D} in (16) to be $\mathcal{D} = \{z : |z| \geq y\}$ and define

$$Z_N(y) := Z_N(\{z : |z| \geq y\}). \quad (54)$$

The probability distribution for the eigenvalue with the smallest modulus z_{\min} is then given by

$$F_N(y) := \mathbb{P}_N(|z_{\min}| \geq y) = \frac{Z_N(y)}{Z_N(0)} = \prod_{n=0}^{N-1} \frac{h_n(y)}{h_n(0)}, \quad (55)$$

where now $h_n(y)$ is given through

$$\int e^{-NV(z)} \mathbb{I}_{\{|z| \geq y\}} p_n(z; y) \overline{p_m(z; y)} d^2z = h_n(y) \delta_{nm}. \quad (56)$$

We want to evaluate the above expression and scale it around the finite inner edge of the support at $|z| = a_- > 0$. We have

$$h_n(y) = 2\pi \int_y^\infty r^{2n+1} e^{-NV(r)} dr = 2\pi \int_y^\infty e^{Nf(r)} dr \quad (57)$$

with $f(r) = (2n+1)/N \log r - V(r)$. Then performing a Taylor expansion and following a similar strategy as in the previous section results in

$$\frac{h_n(y)}{h_n(0)} = \frac{1 + \operatorname{erf}(\sqrt{Nb}(r_0 - y))}{1 + \operatorname{erf}(\sqrt{Nb}r_0)}, \quad (58)$$

where again $b \equiv |f''(r_0)/2|$ and r_0 is given as a solution of (43). Thus in the large N limit one has again (28), i.e.,

$$F_N(y) = N \int_0^1 g(\xi; y) d\xi + \frac{1}{2} [g(1; y) - g(0; y)] + \mathcal{O}(N^{-1}), \quad (59)$$

but where now

$$g(\xi; y) \equiv \log \left(\frac{1 + \operatorname{erf}(\sqrt{Nb}(r_0(\xi) - y))}{1 + \operatorname{erf}(\sqrt{Nb}r_0(\xi))} \right). \quad (60)$$

which using a reasoning similar to the one in the previous two sections yields

$$N \int_0^1 g(\xi; y) d\xi \sim -\frac{N}{2\sqrt{\pi}} \int_0^1 \frac{e^{-Nb(r_0(\xi) - y)^2}}{\sqrt{Nb}(r_0(\xi) - y)} d\xi, \quad N \rightarrow \infty. \quad (61)$$

In fact, the argument is now even simpler since $r_0(\xi)$ stays finite and does not go to zero as ξ goes to zero. This is assured by the requirement that $a_- > 0$.

Now we scale y around the inner edge of the support, a_- , in an analogous way as for the outer edge

$$y = a_- - (Nb)^{-1/2} \left(\alpha_N + \frac{Y}{2\alpha_N} \right). \quad (62)$$

Note that as N becomes large y approaches a_- . Moreover, $r_0(\xi)$ with $\xi = 0$ also approaches a_- as N becomes large, as is clear when comparing (10) with (43). This implies that the leading order contribution of (61) now comes from $\xi = 0$. This suggests the following change of variables in (61)

$$\xi = \gamma_- \frac{(Nb)^{-1/2}}{2\Psi(N)} X. \quad (63)$$

Now expanding $r_0(\xi)$ around $\xi = 0$ gives

$$\begin{aligned} r_0(\xi) &= r_0(0) + dr_0(\xi)/d\xi|_{\xi=0} \xi + \mathcal{O}(\xi^2) \\ &= a_- + \delta_- \xi + \mathcal{O}(\xi^2), \end{aligned} \quad (64)$$

where δ_- is given by

$$\delta_- \equiv dr_0(\xi)/d\xi|_{\xi=0} = \frac{2}{a_- V''(a_-)}. \quad (65)$$

We notice that since γ_- is finite and the potential V is convex, we have $\delta_- > 0$. We therefore obtain

$$\sqrt{Nb}(r_0(\xi) - y) = \alpha_N + \frac{1}{2\alpha_N} (\gamma_- \delta_- X + Y) + \mathcal{O}(N^{-1/2} \alpha_N^{-1}), \quad (66)$$

which by choosing

$$\gamma_- = \frac{1}{\delta_-} = \frac{a_- V''(a_-)}{2} \quad (67)$$

becomes

$$\sqrt{Nb}(r_0(\xi) - y) = \alpha_N + \frac{1}{2\alpha_N} (X + Y) + \mathcal{O}(N^{-1/2} \alpha_N^{-1}). \quad (68)$$

Following the reasoning from the previous section we thus get

$$F(Y) = \lim_{N \rightarrow \infty} \mathbb{P}_N \left(|z_{\min}| \geq a_- - (Nb)^{-1/2} \left[\alpha_N + \frac{Y}{2\alpha_N} \right] \right) = e^{-e^{-Y}}, \quad (69)$$

where α_N is given in (38). This shows universality by proving convergence of the rescaled distribution of the eigenvalue with smallest modulus to a Gumbel distribution. The derivation assumes that the potential fulfills the condition (45) or (46) and that the inner radius $a_- > 0$, i.e., the support of the eigenvalue density has topology of a ring.

7. Discussion

The celebrated Tracy-Widom distribution provides the extreme value statistics for Hermitian random matrices. For non-Hermitian matrices the situation is somewhat easier as eigenvalues are only weakly correlated and the extreme value statistics is given by the much simpler Gumbel distribution. However, due to the lack of symmetries, non-Hermitian random matrices are in general harder to deal with. A special case are normal random matrices which are non-Hermitian and at the same time allow for a Coulomb gas formulation for general potential.

In this work we investigate the extreme value statistics of normal random matrices and 2D Coulomb gases for general radially symmetric potentials. This is done by extending the orthogonal polynomial approach, introduced in [21] for Hermitian matrices, to normal random matrices and 2D Coulomb gases. We first analyse the simplest case of Gaussian normal random matrices and show convergence of the eigenvalue with largest modulus rescaled around the outer edge of the eigenvalue support to a Gumbel distribution. One strength of this approach lies in the fact that it immediately generalises to an arbitrary potential $V = V(r)$ with radial symmetry which satisfies the condition (8) together with the condition (45) or (46). We use this to show universality of the distribution of the eigenvalue with largest modulus when rescaled around the outer edge of the eigenvalue support. This provides an alternative, simplified derivation of results presented in [13, 14]. In addition, it is shown that the approach presented here also generalised to compute convergence of the distribution of the eigenvalue with smallest modulus rescaled around the inner edge of the eigenvalue support with topology of an annulus.

The here presented approach can also be used to obtain finite N results. Firstly, for Ginibre matrices, the expression (24) can easily be evaluated numerically for finite N . This is done in the comparison presented in Figure 3 where we plot the numerically evaluated cumulative probability function (24) against the empirical cumulative probability function obtained from samples of Ginibre matrices for finite N . Secondly, the expansion (28), and the analogous expression for general potential, can in principle be used to obtain an analytical expression of the next-to-leading-order term. We leave the details of such an analysis for future work. Note that this calculation follows closely the standard finite N expansion of the free energy of Hermitian random matrices using orthogonal polynomials (see for example Section 2.3 of [27]). Another interesting direction of future research is to explore the extreme value statistics of the eigenvalue with smallest modulus at the transition when the inner radius of the eigenvalue support goes to zero as N goes to infinity. This transition was analysed

in [24] in the context of the mean radial displacement. It would be interesting to see whether the extreme value statistics can be tuned to a different universality class in this transition by using a double scaling limit in which a_- goes to zero as N becomes large.

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Appendix A. Numerical methods

Appendix A.1. Ginibre matrices

For the Gaussian case we can generate samples from the joint eigenvalues density by first creating a complex matrix of random elements

$$M_{ij} = \frac{1}{\sqrt{2N}} (A_{ij} + iB_{ij}), \quad \text{with } A_{ij}, B_{ij} \sim \mathcal{N}(0, 1) \quad (\text{A.1})$$

and then use standard techniques to numerically diagonalise the matrix.

As a concrete example in R one can write the following code, which generates m samples of the matrix M_{ij} , diagonalises it, extracts the eigenvalue with the largest modulus and stores it in an array \vec{y} .

```
y <- rep(NULL, m)
for (i in 1:m){
  M = matrix(complex(real=rnorm(N*N), imaginary=rnorm(N*N)), ncol=N)
  y[i] = sort(Mod(eigen(M, symmetric=FALSE,
    only.values = TRUE)$values)/sqrt(N*2))[N] }
```

Appendix A.2. Monte-Carlo methods for normal random matrices and Coulomb gases

To obtain samples of the eigenvalues \vec{z} from the joint eigenvalue density (4) for a general (radially symmetric) potential we employ Monte-Carlo techniques. To do so we decompose $\vec{z} = \vec{x} + i\vec{y}$ and initialise the N elements of \vec{x} and \vec{y} with uniform random numbers in the interval $(-1, 1)$. Then we iterate over the following steps. First one perturbs a given randomly chosen element of \vec{x} and \vec{y} each by a Gaussian random number, multiplied by a scaling η/N where η is a number of $\mathcal{O}(1)$. Given the perturbed vector \vec{z}' we then evaluate the difference in energy in the Boltzmann weight (5),

$$\Delta E = N^2 (S_{\text{eff}}(\vec{z}) - S_{\text{eff}}(\vec{z}')). \quad (\text{A.2})$$

If the new configuration has lower energy, i.e., $\Delta E > 0$, we change \vec{z} to \vec{z}' . If the new configuration has larger energy we still allow to change \vec{z} to \vec{z}' if $\exp(\Delta E)$ is larger than a random uniform number u in $(0, 1)$. This is the standard Metropolis step. The above iteration is repeated m times where m is usually chosen $\mathcal{O}(N^3)$.

For completeness we provide the pseudo-code of the algorithm below.

```

1: procedure MONTE CARLO SIMULATION FOR EIGENVALUE DISTRIBUTION
2:   initialise:  $N, m, \eta$  and the function  $V(r)$ 
3:   initialise:  $\vec{x} \leftarrow \text{uniform}(-1, 1)$ 
4:   initialise:  $\vec{y} \leftarrow \text{uniform}(-1, 1)$ 
5:   for iteration in  $1, \dots, m$  do
6:     // perturb one randomly chosen element of  $\vec{x}$  and  $\vec{y}$ :
7:      $k \leftarrow \text{choose}([1, \dots, N])$ 
8:      $x'[k] \leftarrow x[k] + \eta\delta_x/N$  with  $\delta_x \sim \text{normal}(0, 1)$ 
9:      $y'[k] \leftarrow y[k] + \eta\delta_y/N$  with  $\delta_y \sim \text{normal}(0, 1)$ 
10:    // Calculate energy difference for  $\vec{z} = \vec{x} + i\vec{y}$  and  $\vec{z}' = \vec{x}' + i\vec{y}'$ :
11:     $\Delta E = N^2(S_{\text{eff}}(\vec{z}) - S_{\text{eff}}(\vec{z}'))$ 
12:    // sample a uniform number and do the Metropolis step:
13:     $u \sim \text{uniform}(0, 1)$ 
14:    if  $\exp(\Delta E) > u$  then
15:       $\vec{x} \leftarrow \vec{x}'$ 
16:       $\vec{y} \leftarrow \vec{y}'$ 
17:    end if
18:  end for
19: end procedure

```

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